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Surface waves in basins of variable depth

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The linearized boundary-value problem for surface waves of frequency ω in a closed basin of variable depth is reduced to a non-self-adjoint partial differential equation in the plane of the free surface. The corresponding variational form (which does not provide a definite upper or lower bound) for the eigenvalue $\kappa = \omega^2/g$ is constructed. A self-adjoint partial differential equation, for which the variational form is the Rayleigh quotient (which provides an upper bound to κ), also is constructed; it offers significant advantages vis-à-vis the non-self-adjoint formulation, but at the expense of a more complicated operator. Three relatively simple variational approximations are constructed, two for a class of basins with sloping sides and the third for basins for which the variation of the depth relative to its mean is small. These general results are illustrated by comparison with Rayleigh's (1899) results for a semicircular channel, Sen's (1927) inverse results for a family of circular basins, and Lamb's (1932) results for a shallow circular paraboloid. The eigenvalue for the dominant mode in the paraboloid is determined through $O(\delta^5)$, where δ = depth/radius.

1. Introduction

Surface waves in water of variable depth, known as 'seiches' in lakes and inland seas (Hutchinson 1975), receive only limited attention in the standard treatises (Lamb 1932; Wehausen & Laitone 1960). Various semi-empirical formulae and numerical methods (Chrystal 1905; Proudman 1915) are available for the estimation of their periods, and it now is possible, using discrete-grid or discrete-element methods, to obtain numerical solutions for basins of any realistic shape; nevertheless, analytical solutions remain important both as checks for numerical algorithms and as indicators of parametric trends.

A few exact solutions (of the linearized equations of motion) are available. Kirchhoff (1879) determined the complete set of modes for a channel of triangular cross-section, the sides of which form angles of $m\pi/n$ with the horizontal, and Vint (1923) obtained similar results for an inverted, four-sided pyramid, the sides of which form angles of $\frac{1}{4}\pi$ with the horizontal. Lamb (§193)† determined the complete set of modes for a shallow circular paraboloid, and Goldsbrough (1930) carried out the corresponding calculation for a shallow elliptical paraboloid. Sen (1927) and Storchi (1949, 1952) determined basin shapes for assumed solutions, an inverse procedure that yields only a single mode for a particular basin (typically the dominant mode, which is the one of primary interest in practice) but is efficient and appears to merit further exploitation (see §3). Lundberg (1984) obtained second approximations (first approximations being provided by shallow-water theory) for the dominant antisymmetric and symmetric modes in a parabolic channel.

† This and subsequent references to Lamb are to sections in the 6th edition of *Hydrodynamics* (Lamb 1932).

Approximate solutions may be obtained by Rayleigh's (1873) method, which Lamb (§259) applied to obtain the approximation $\omega = 1.169(g/a)^{\frac{1}{2}}$ to the natural frequency of the dominant mode in a semicircular channel of radius a. Rayleigh subsequently (1899) obtained the second approximation $\omega = 1.1644(g/a)^{\frac{1}{2}}$, the proximity of which to Lamb's first approximation suggests the effectiveness of the method and illustrates Rayleigh's principle that successive approximations are increasingly accurate upper bounds to the true frequency.

The preceding solutions neglect nonlinear terms in the equations of motion. Ball (1963) used an inverse procedure to obtain exact solutions of the nonlinear, shallow-water equations for the dominant modes in a parabolic channel and a paraboloidal basin.

Consider a basin of maximum depth d, horizontal scale a, free surface S, and lateral boundary ∂S. The equations governing small, irrotational motion of an inviscid, incompressible fluid in the basin may, in principle, be reduced to a linear, homogeneous partial differential equation for either the free-surface displacement or the velocity potential in S, the solutions of which, subject to a kinematic boundary condition on ∂S, yield an infinite, discrete set of eigenvalues for the natural frequencies of the free oscillations. If

$$\delta = d/a \tag{1.1}$$

is sufficiently small this partial differential equation may be approximated by (Lamb, §193)

$$\nabla \cdot (h \nabla \zeta) + \kappa \zeta = 0 \quad (\delta \to 0), \tag{1.2}$$

where ∇ is the gradient operator in a horizontal plane, $h = h(\boldsymbol{x})$ is the local depth, $\zeta = \zeta(\boldsymbol{x}, t)$ is the free-surface displacement, which is assumed to be harmonic in t with angular frequency ω,

$$\kappa = \omega^2/g \tag{1.3}$$

(an inverse length) is the eigenvalue, $\boldsymbol{x} = (x, y)$ is the horizontal coordinate, and the implicit error factor is $1 + O(\delta^2)$. The kinematic boundary condition is

$$h(\boldsymbol{n} \cdot \nabla \zeta) = 0 \quad (\boldsymbol{x} \text{ on } \partial S), \tag{1.4}$$

where $\boldsymbol{n}$ is the normal to ∂S.

The smallest of the eigenvalues of (1.2) and (1.4) may be approximated by substituting a suitable approximation to ζ into the Rayleigh quotient

$$\kappa = \langle h(\nabla \zeta)^2 \rangle / \langle \zeta^2 \rangle, \tag{1.5}$$

where, here and subsequently, $\langle\ \rangle$ signifies an average over S. It follows from Rayleigh's principle, or from a direct construction of the first and second variations, that the quadratic functional (1.5) is a minimum with respect to first-order variations of ζ about the true solution of (1.2) and (1.4). (Higher eigenvalues also can be approximated by (1.5) if the trial function is orthogonal to each of the lower modes, but practical application typically is limited to the smallest eigenvalue or, in the case of a symmetrical basin, the smallest eigenvalue for each of the antisymmetric and symmetric motions.)

I consider here the generalization of (1.2) and (1.5) for arbitrary depth with the primary aim of extending Lamb's shallow-water formulation to small but finite δ. In §§2–4, I develop non-self-adjoint counterparts of (1.2) and (1.5) with the velocity potential at the free surface as the dependent variable and, following Sen (1927), obtain exact solutions for the dominant modes in one-parameter families of channels and circular and elliptical basins and for the dominant axisymmetric modes in a second family of circular basins. In §§5 and 6, I develop a self-adjoint formulation and obtain explicit counterparts of (1.2) and (1.5) with error factors of $1+O(\delta^4)$, which should suffice for the description of the effects of small but finite depth in most applications. I develop a matrix formulation, based on the complete set of eigenfunctions for the Helmholtz equation in S, in the Appendix. This last development provides a systematic procedure for solving the boundary-value problem to any desired accuracy, but (since direct numerical solutions are likely to be more efficient for high-order approximations) its primary value is its elucidation of the formulations in §§2–6.

Perhaps the most useful results of the present paper are the variational approximations:

$$\kappa = \langle h \rangle / \langle x^2 + \tfrac{1}{2}h^2 \rangle \tag{1.6}$$

for the dominant mode in a basin for which S is symmetrical with respect to $x = 0$ and h vanishes smoothly at ∂S (i.e. the basin has sloping sides); the successive approximations

$$\kappa = k \tanh (k \langle h \rangle) \tag{1.7}$$

and
$$\kappa = k \tanh (k \langle h \rangle) + [\langle \hat{h} (\nabla \Phi)^2 \rangle / \langle \Phi^2 \rangle] \operatorname{sech}^2 (k \langle h \rangle) \tag{1.8}$$

for the dominant mode in a basin for which $\hat{h}$, the variation of h from its mean value $\langle h \rangle$, is small compared with $\langle h \rangle$, and Φ and k are determined by the solution of

$$(\nabla^2 + k^2)\,\Phi = 0, \quad \boldsymbol{n}\cdot\nabla\Phi = 0 \quad \text{on } \partial S. \tag{1.9a,b}$$

The latter solution may be determined approximately by minimizing

$$k^2 = \langle (\nabla \Phi)^2 \rangle / \langle \Phi^2 \rangle. \tag{1.10}$$

2. The boundary-value problem

The assumption of irrotational flow of an inviscid, incompressible fluid in the rigid basin $z = -h(\boldsymbol{x})$ with the free surface $z = \zeta$ implies the existence of a velocity potential ϕ that satisfies: Laplace's equation, which we write in the form

$$(\partial_z^2 - \ell^2)\,\phi = 0, \quad \ell^2 \equiv -\nabla^2 \equiv -(\partial_x^2 + \partial_y^2), \tag{2.1a,b}$$

where z is measured positive up along a vertical axis and ∇^2 is the Laplacian operator in a horizontal plane; the boundary condition

$$\phi_z + \nabla h \cdot \nabla \phi = 0 \quad (z = -h); \tag{2.2}$$

the (linearized) free-surface conditions

$$\phi_z = \zeta_t, \quad \phi_t + g\zeta = 0 \quad (z = 0), \tag{2.3a,b}$$

which, on the assumption of harmonic motion of frequency ω, may be combined to obtain

$$\phi_z = \kappa\phi \quad (z = 0), \tag{2.4}$$

where $\kappa = \omega^2/g$. It is implicit that $\omega \gg f$, where f is the Coriolis frequency – or, equivalently, that the period is small compared with 12 h.

We pose the solution of (2.1a) in the form (Sen 1927)

$$\phi = \{\cosh \ell z\, \Phi(\boldsymbol{x}) + \ell^{-1} \sinh \ell z\, \Psi(\boldsymbol{x})\} \cos (\omega t + \alpha), \tag{2.5}$$

where the operators $\cosh \ell z$ and $\ell^{-1} \sinh \ell z$ are defined by their power-series expansions together with (2.1b), and α is an arbitrary phase constant.† Invoking (2.2) and (2.4), we obtain

$$\{-\ell \sinh \ell h+\cosh \ell h(\nabla h\cdot\nabla)\}\Phi+\{\cosh \ell h-\ell^{-1}\sinh \ell h(\nabla h\cdot\nabla)\}\Psi=0, \quad (2.6)$$

where, here and in (2.8) below, ℓ^2 operates only on Φ and Ψ (but not on h), and

$$\Psi=\kappa\Phi. \quad (2.7)$$

Substituting (2.7) into (2.6) and invoking the operational identities (2.1b) and $\nabla\cdot(U\nabla V)=U\nabla^2V+\nabla U\cdot\nabla V$, we obtain

$$\nabla\cdot\left[\left\{\frac{\sinh \ell h}{\ell}+\kappa\left(\frac{1-\cosh \ell h}{\ell^2}\right)\right\}\nabla\Phi\right]+\kappa\Phi=0, \quad (2.8)$$

wherein $\nabla\cdot$ operates on both Φ and h (whereas ℓ^2 operates only on Φ). The corresponding boundary condition is (essentially (1.4))

$$h(\boldsymbol{n}\cdot\nabla\Phi)=0 \quad (\boldsymbol{x} \text{ on } \partial S). \quad (2.9)$$

Conservation of mass requires $\langle\zeta\rangle=0$, which, through (2.3b), implies the constraint

$$\langle\Phi\rangle=0. \quad (2.10)$$

Letting $h\to 0$ in (2.8) with $\kappa=O(h)$, we recover (1.2). Higher approximations to (2.8) for quasi-shallow basins may be obtained by expanding the operators in powers of h, but if only the next (beyond the shallow-water) approximation is required it is preferable to invoke the self-adjoint form (5.4).

If h is constant (Lamb, §257) (2.8) and (2.9) may be reduced to (1.9a,b) and

$$\kappa=k\tanh kh \quad (h=\text{constant}), \quad (2.11)$$

where k^2 now is one of the family of eigenvalues determined by (1.9b).

3. Inverse solutions

Exact solutions of (2.8) may be obtained by positing Φ as a polynomial of degree N in x and y, for which $\nabla^{2n}\nabla\Phi=0$ if $n>\frac{1}{2}(N-1)$, and solving for h. The simplest case is (Sen 1927)

$$\Phi=(gA/\omega a)x, \quad (3.1)$$

for which the oscillating free surface, $\zeta=A(x/a)\sin(\omega t+\alpha)$, remains plane and (2.8) reduces to

$$\partial_x(h-\tfrac{1}{2}\kappa h^2)+\kappa x=0. \quad (3.2)$$

Integrating (3.2) and determining the additive function of y and the eigenvalue κ through the assumptions that

$$h=0 \quad \text{on } x=\pm X(y), \quad X(0)\equiv a, \quad h(0)\equiv d, \quad (3.3a,b,c)$$

we obtain $$h=l-[l^2+x^2-X^2(y)]^{\frac{1}{2}}, \quad l\equiv\frac{1}{\kappa}=\tfrac{1}{2}a(\delta+\delta^{-1}) \quad (0<\delta\leqslant 1) \quad (3.4a,b)$$

† Solutions of the wave equation in the form (2.5) appear in Rayleigh's *Theory of Sound* (1878), where they are implicitly attributed to Poisson (1820). Sen (1927) gives (2.5) with Ψ replaced therein by (2.7), but he does not obtain the partial differential equation (2.8) for Φ and evidently overlooks the constraint (2.10). The convergence of the operational expansions may be tested in particular cases, for many of which the expansions are finite.

((3.4) yields $h = a^2/d$ at $x = y = 0$ if $\delta > 1$). Combining (1.1), (1.3), and (3.4*b*), we obtain

$$\frac{\omega^2 a}{g} = \frac{2\delta}{(1+\delta^2)} \quad (0 < \delta \leqslant 1). \tag{3.5}$$

Interesting choices of $X(y)$ are

$$X = a, \quad X = (a^2-y^2)^{\frac{1}{2}}, \quad X = a\left[1-\left(\frac{y}{b}\right)^2\right]^{\frac{1}{2}}. \tag{3.6a, b, c}$$

$X = a$ yields a channel with a hyperbolic profile that varies from a parabola as $\delta \downarrow 0$ to a right triangle with sides inclined at 45° to the vertical (a particular case of Kirchhoff's problem; see Lamb, §258) as $\delta \uparrow 1$. The choice (3.6*b*) yields a family of circular basins that vary from a paraboloid as $\delta \downarrow 0$ to a right-angled cone as $\delta \uparrow 1$. (The solution for the paraboloid agrees with that given by Lamb (§193) in the shallow-water approximation; however, Lamb gives the solution for all modes, whereas the present solution gives only the dominant mode.) The choice (3.6*c*) yields a corresponding family of elliptical basins. More generally, (3.1) and (3.5) provide the solution for the dominant mode in any basin that is symmetrical with respect to its shorter axis (it also provides a solution for a basin that is symmetrical with respect to its longer axis, but this solution typically does not describe the dominant mode) and has a depth profile of the form (3.4*a*); in particular,

$$h = \frac{d}{a^2}[X^2(y)-x^2], \quad \omega^2 = \frac{2gd}{a^2} \quad (\delta \downarrow 0) \tag{3.7a, b}$$

for a shallow basin.

We obtain an axisymmetric solution that satisfies (2.10)† by positing

$$\Phi = \frac{gA}{\omega}\left[1-2\left(\frac{r}{a}\right)^2\right] \quad (0 \leqslant r \leqslant a) \tag{3.8}$$

and proceeding as above. The end results are

$$h = l-[l^2+\tfrac{1}{4}(r^2-a^2)]^{\frac{1}{2}}, \quad l = \tfrac{1}{2}a(\delta+\tfrac{1}{4}\delta^{-1}) \quad (0 < \delta \leqslant \tfrac{1}{2}), \tag{3.9a, b}$$

and

$$\frac{\omega^2 a}{g} = \frac{2\delta}{\frac{1}{4}+\delta^2} \quad (0 < \delta \leqslant \tfrac{1}{2}). \tag{3.10}$$

The circular basin described by (3.9) varies from a paraboloid as $\delta \downarrow 0$ to a cone as $\delta \uparrow \frac{1}{2}$, but the family differs from that of the preceding paragraph, and the eigenvalues (3.5) and (3.10) do not belong to the same basin except in the limit $\delta \downarrow 0$, in which both correspond to the paraboloid $h/d = 1-(r/a)^2$ and reduce to $\omega^2 a^2/gd = 2$ and 8, respectively, in agreement with Lamb's (§193) results.

Additional solutions may be generated by positing higher-order polynomials for Φ; e.g.

$$\Phi = r^m(A_0+A_1 r^2+A_2 r^4+\ldots)\cos m\theta \quad (m = 1, 2, \ldots) \tag{3.11}$$

for a circular basin. However, as already noted, this procedure does not lead to a sequence of modes for a particular basin, and only the preceding results are likely to be of practical interest.

† Sen (1927) evidently overlooks the constraint (2.10), in consequence of which his axisymmetric solutions are physically inadmissable. This oversight does not affect his solutions based on (3.1), which satisfies (2.10) by virtue of antisymmetry.

4. Variational approximations

Multiplying (2.8) through by a function Φ^* that satisfies (2.9) and (2.10), integrating over S, transforming the integrals of the divergence terms with the aid of Green's theorem, and requiring Φ to satisfy (2.9), we obtain

$$\kappa = \frac{\langle \nabla \Phi^* \cdot (k^{-1} \sinh kh \nabla \Phi)\rangle}{\langle \Phi^* \Phi \rangle + \langle \nabla \Phi^* \cdot \{k^{-2}(\cosh kh - 1) \nabla \Phi\}\rangle}, \tag{4.1}$$

which is invariant under scale transformations of Φ and Φ^* and (Finlayson 1972) stationary with respect to joint variations of Φ about the solution of (2.8) and of Φ^* about the adjoint of (2.8). We remark that (4.1) reduces to (1.5) in the limit $h \to 0$ with $\Phi^* = \Phi$; however, it is only in this limit that (2.8) is self-adjoint and that (4.1) is the Rayleigh quotient (cf. (6.3)). Except in this limit, we cannot assert that (4.1), *qua* variational approximation, provides either an upper or a lower bound to the true value of κ.

Perhaps the simplest trial function for the dominant mode in a basin for which $h = 0$ along ∂S (which condition ensures the satisfaction of (2.9)) and for which S is at least approximately symmetrical is given by (3.1). In the present application, x is directed across the axis of a two-dimensional channel or along the longer axis of a three-dimensional basin (it also may be directed along the shorter axis to obtain one of the higher eigenvalues). Substituting $\Phi = \Phi^* = x$ into (4.1) and expanding the operators (note that $\nabla^{2n} x = 0$ for $n > 0$), we obtain

$$\kappa = \langle h \rangle / \langle x^2 + \tfrac{1}{2} h^2 \rangle. \tag{4.2}$$

Consider, for example, a semicircular cylinder of radius a, for which the substitution of

$$h = (a^2 - x^2)^{\frac{1}{2}} \tag{4.3}$$

into (4.2) yields $\omega (a/g)^{\frac{1}{2}} = 1.085$. This compares with Lamb's (§259) approximation, $\omega(a/g)^{\frac{1}{2}} = 1.169$, obtained through the same approximation to Φ in the true Rayleigh quotient. It is clear from Rayleigh's second approximation, $\omega(a/g)^{\frac{1}{2}} = 1.1644$, that the present approximation is less accurate than that provided by the Rayleigh quotient and that, unlike the latter, it does not provide an upper bound to the true result; on the other hand, its calculation is manifestly simpler.

As a second example, consider the circular paraboloid

$$h = d \left[1 - \left(\frac{r}{a} \right)^2 \right], \tag{4.4}$$

for which (4.2) is exact in the limit $\delta \downarrow 0$ (see above) and yields

$$\frac{\omega^2 a^2}{2gd} = (1 + \tfrac{2}{3}\delta^2)^{-1} \tag{4.5a}$$

$$= 1 - \tfrac{2}{3}\delta^2 + \tfrac{4}{9}\delta^4 + \dots \tag{4.5b}$$

for finite δ. The coefficient $-\frac{2}{3}$ in (4.5b) proves to be exact† (see §6), whilst the coefficient $\frac{4}{9}$ compares with the exact value $\frac{17}{27}$.

The counterpart of (4.5a) for a parabolic channel, for which r is replaced by x in (4.4), is $\omega^2 a^2/2gd = (1 + \frac{4}{5}\delta^2)^{-1} \approx 1 - \frac{4}{5}\delta^2$, in agreement with Lundberg (1984).

† In this example, in which the error in the trial function is $O(\delta^2)$, the error in the variational approximation is $O(\delta^4)$, just as with the Rayleigh quotient (see §6). This appears to follow from the fact that (2.8) is self-adjoint for that class of functions for which $\nabla^2 \Phi = 0$, which includes (3.1).

The dominant solution of (1.9) provides an alternative trial function, the substitution of which for both Φ and Φ^* in (4.1), together with the identities (which hold only for solutions of (1.9))

$$\not{k}^2\Phi = -\nabla^2\Phi = k^2\Phi, \quad \langle(\nabla\Phi)^2\rangle = k^2\langle\Phi^2\rangle, \tag{4.6a, b}$$

yields
$$\kappa = k\langle(\nabla\Phi)^2 \sinh kh\rangle/\langle(\nabla\Phi)^2 \cosh kh\rangle. \tag{4.7}$$

The approximation (4.7) is especially appropriate if the variation of h from its mean value $\langle h\rangle$ is small. Substituting

$$h = \langle h\rangle + \hat{h} \quad (|\hat{h}| \ll \langle h\rangle) \tag{4.8}$$

into (4.7), expanding the hyperbolic functions about $h = \langle h\rangle$, and invoking (4.6*b*), we obtain the successive approximations

$$\kappa = k \tanh(k\langle h\rangle) + O(k^2\hat{h}) \tag{4.9}$$

and
$$\kappa = k \tanh(k\langle h\rangle) + [\langle\hat{h}(\nabla\Phi)^2\rangle/\langle\Phi^2\rangle] \operatorname{sech}^2(k\langle h\rangle) + O(k^3\hat{h}^2). \tag{4.10}$$

The latter approximation is identical with the corresponding approximation to the Rayleigh quotient (see Appendix) and therefore is an upper bound within $1 + O(k^2\hat{h}^2)$.

The eigenvalue k and the corresponding eigenfunction Φ in the preceding approximations may be determined from the variational form (1.10). For example, the trial functions

$$\Phi = a^2x - \tfrac{1}{3}x^3, \quad \Phi = (a^2r - \tfrac{1}{3}r^3)\cos\theta \tag{4.11a, b}$$

to the dominant modes in a channel of width $2a$ and a circle of radius a, respectively, yield $ka = 1.5718$ and 1.8423, which differ from the corresponding exact results, $\tfrac{1}{2}\pi$ and 1.8412, by $0.06\,\%$. But we emphasize that these are the errors in (4.11*a, b*) *qua* approximations to the solution of (1.9), not (2.8).

Consider, as a first example, the semicircular cylinder. Substituting (4.3), (4.11*a*) and $ka = 1.572$ into (4.9)/(4.10), we obtain $\omega(a/g)^{\frac{1}{2}} = 1.152/1.193$, which is $1.1\,\%$ below/$2.5\,\%$ above Rayleigh's second approximation (see above).

As a second example, consider the family of circular basins described by (3.4) and (3.6*b*),

$$h = l - (l^2 - a^2 + r^2)^{\frac{1}{2}}, \quad l = \tfrac{1}{2}a(\delta + \delta^{-1}) \quad (0 < \delta \leqslant 1). \tag{4.12a, b}$$

Substituting (4.11*b*) and (4.12) into (4.10) and carrying out the integrations, we obtain

$$\frac{\omega^2 a}{g} = ka \tanh(k\langle h\rangle) + \tfrac{14}{33}\delta\left(1 + \frac{\delta^2}{21} - \frac{\delta^4}{5} - \frac{\delta^6}{49}\right) \operatorname{sech}^2(k\langle h\rangle), \tag{4.13a}$$

$$\langle h\rangle = \tfrac{1}{2}d(1 - \tfrac{1}{3}\delta^2) \quad (ka = 1.8423), \tag{4.13b}$$

which is compared with the corresponding approximation (4.9) and the exact result (3.5) in figure 1. The limiting results for the paraboloid ($\delta \downarrow 0$) and the cone ($\delta = 1$) are $\omega a/(2gd)^{\frac{1}{2}} = 1.030$ and $\omega(a/g)^{\frac{1}{2}} = 1.120$, respectively, which compare with the exact limits of 1. The corresponding approximations provided by (4.9) are 0.921 and 1.001, respectively (it is evident from figure 1 that the proximity of the last approximation to the exact result is somewhat fortuitous). It should be noted that $|\hat{h}|/\langle h\rangle$ is not small for the family (4.12) (it has a maximum of 2 for the cone), which therefore provides a rather severe test of the approximation (4.10).

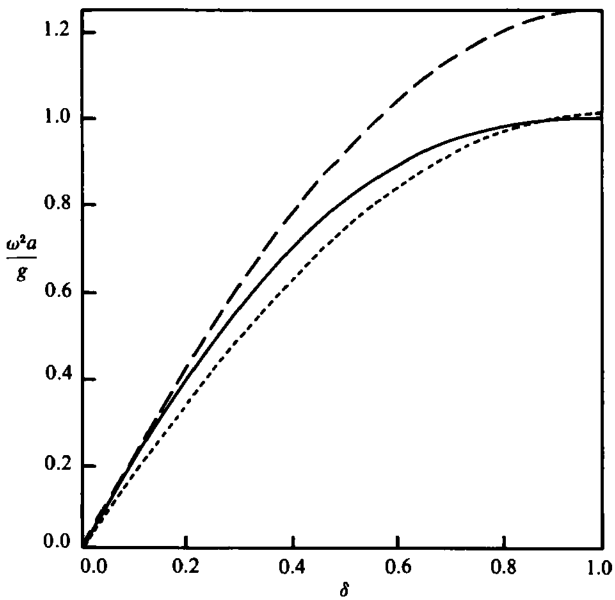

FIGURE 1. The approximations (4.9) (----) and (4.3) (---) compared with the exact result (3.5) (——) for the family of circular basins (4.12).

5. Self-adjoint formulation

The partial differential equation

$$\mathscr{L}\Phi = \kappa\Phi, \tag{5.1}$$

where the operator $\mathscr{L}$ is defined by

$$\Psi = \mathscr{L}\Phi, \tag{5.2}$$

may be obtained by solving (2.6) for Ψ and then invoking (2.7). We remark that in this derivation, in contrast to that of (2.8), $\mathscr{L}$ is determined by kinematics alone, i.e. by Laplace's equation and the kinematical boundary condition (2.2), after which the dynamical boundary condition (2.4) implies (5.1), which identifies κ as an eigenvalue of $\mathscr{L}$.

Expanding the operators in (2.6) and solving for Ψ, we obtain

$$\mathscr{L} = \left\{1 + \sum_{n=1}^{\infty} \frac{(-)^n}{(2n)!} \nabla\cdot h^{2n}\,\nabla^{2n-1}\right\}^{-1} \sum_{n=0}^{\infty} \frac{(-)^{n-1}}{(2n+1)!} \nabla\cdot h^{2n+1}\,\nabla^{2n+1}, \tag{5.3}$$

which may be expanded in powers of δ.† The first approximation to (5.1), retaining only the leading term in $\mathscr{L}$, is essentially (1.2). The second approximation, for which the implicit error factor is $1+O(\delta^4)$, is

$$\nabla\cdot\{h\,\nabla\Phi + \tfrac{1}{2}h^2\,\nabla\nabla\cdot(h\,\nabla\Phi) - \tfrac{1}{6}h^3\,\nabla^3\Phi\} + \kappa\Phi = 0. \tag{5.4}$$

It can be demonstrated that (5.4) is self-adjoint. I have not proved that $\mathscr{L}$ is self-adjoint at any order, but this appears to follow indirectly from Rayleigh's principle (see below).

† The convergence of this expansion may be tested for a particular operand, but the question of convergence to the true solution for specified h remains open. The convergence of the corresponding expansion of the eigenvalue for the self-adjoint operator is guaranteed by Rayleigh's principle.

6. The Rayleigh quotient

The mean kinetic and potential energies of the fluid motion are given by

$$\langle T\rangle = \tfrac{1}{2}\rho S\langle\overline{\phi\phi_z}\rangle = \tfrac{1}{4}\rho S\langle\Phi\Psi\rangle \tag{6.1}$$

and

$$\langle V\rangle = \tfrac{1}{2}\rho g S\langle\overline{\zeta^2}\rangle = \tfrac{1}{4}\rho g^{-1}\omega^2 S\langle\Phi^2\rangle, \tag{6.2}$$

where (2.5) and (2.3b) have been invoked and the overbar implies a temporal average. Equating $\langle T\rangle$ and $\langle V\rangle$ and invoking $\omega^2/g = \kappa$ and $\Psi = \mathscr{L}\Phi$, we obtain

$$\kappa = \langle\Phi\mathscr{L}\Phi\rangle/\langle\Phi^2\rangle \tag{6.3}$$

as a generalization of (1.5). It follows from Rayleigh's principle that the quadratic functional in (6.3) is a minimum with respect to first-order variations of Φ about the true solution of (5.1) and (2.9). (The variational form (6.3) also may be derived by multiplying (5.1) by Φ and integrating over S, but it then would remain to identify the result as the Rayleigh quotient.)

Approximating $\mathscr{L}$ as in (5.4) and invoking Green's theorem and (2.9), we obtain

$$\kappa = \langle h\{(\nabla\Phi)^2 - \tfrac{1}{3}(h\nabla^2\Phi)^2 - (h\nabla^2\Phi)(\nabla h\cdot\nabla\Phi) - (\nabla h\cdot\nabla\Phi)^2\}\rangle/\langle\Phi^2\rangle, \tag{6.4}$$

wherein the implicit error factor is $1+O(\delta^4)$.

Adopting the trial function $\Phi = x$ in (6.4), we obtain

$$\kappa = \langle h(1-h_x^2)\rangle/\langle x^2\rangle \tag{6.5}$$

as a counterpart of (4.2), but the error factor now is $1+O(\delta^4,\epsilon^2)$, where ϵ is the relative error in Φ, whereas the error factor (4.2) may be as large as $1+O(\epsilon)$. Retaining terms through $O(h^5)$ in the expansion of $\mathscr{L}$ with the trial function $\Phi = x$, we obtain (after partial integrations via Green's theorem)

$$\kappa = \langle h - hh_x^2 - \tfrac{1}{3}h^3h_x\nabla^2h_x - h^2h_x(\nabla h\cdot\nabla h_x)\rangle/\langle x^2\rangle, \tag{6.6}$$

for which the error factor is $1+O(\delta^6,\epsilon^2)$. We note that these last approximations, unlike (4.2), involve derivatives of h and may fail if h is singular; e.g. (6.5) yields $\kappa = 0$, and (6.6) contains divergent integrals, for (4.3), which is singular at $x = \pm a$.

Consider, for example, the circular paraboloid described by (4.4), the substitution of which into (6.5) and (6.6) yields (cf. (4.5a,b))

$$\frac{\omega^2a^2}{2gd} = 1-\tfrac{2}{3}\delta^2 \tag{6.7a}$$

and

$$\frac{\omega^2a^2}{2gd} = 1-\tfrac{2}{3}\delta^3+\tfrac{2}{3}\delta^4 \tag{6.7b}$$

respectively, in which the coefficient $-\frac{2}{3}$ is exact. It appears from (6.9b) that the coefficient of δ^4 in (6.7b), $\frac{2}{3}$, is closer to the exact result $\frac{17}{27}$ than the coefficient $\frac{4}{9}$ in (4.5b). On the other hand, the approximation (4.5a) appears to be superior to (6.7a).

We proceed to the next variational approximation by substituting

$$\Phi = \left(\frac{gA}{\omega a}\right) r\left[1 + C\delta^2\left(\frac{r}{a}\right)^2 + O(\delta^4)\right]\cos\theta \tag{6.8}$$

into (6.3), retaining the terms through $O(h^5)$ in the expansion of $\mathscr{L}$, and requiring κ to be stationary with respect to the parameter C. The end results are

$$C = -\tfrac{1}{3}+O(\delta^2), \quad \frac{\omega^2 a^2}{2gd} = 1-\tfrac{2}{3}\delta^2+\tfrac{17}{27}\delta^4+O(\delta^6), \tag{6.9a, b}$$

in which the coefficients of δ^2 and δ^4 are exact. These last results may be confirmed by solving (2.8) to the required approximation, which yields (6.9b) and

$$\Phi = \left(\frac{gA}{\omega a}\right) r\left[1-\tfrac{1}{3}\delta^2(1-\tfrac{19}{36}\delta^2)\left(\frac{r}{a}\right)^2+\tfrac{1}{8}\delta^4\left(\frac{r}{a}\right)^4+O(\delta^6)\right]\cos\theta. \tag{6.10}$$

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Appendix. Matrix formulation

A complete set of orthogonal functions in S, $\{\psi_n; k_n\}$, is provided by the solutions of (1.9), which may be normalized to satisfy

$$\langle\psi_m\psi_n\rangle = \delta_{mn}, \quad \langle\nabla\psi_m\cdot\nabla\psi_n\rangle \equiv \langle q_{mn}\rangle = \delta_{mn}k_n^2, \tag{A 1a, b}$$

where δ_{mn} is the Kronecker delta. The solution ψ_0 = constant, which is associated with the eigenvalue $k_0 \equiv 0$, is omitted in consequence of the constraint (2.10).

Substituting the Fourier expansion

$$\Phi = \sum_n A_n\psi_n(\boldsymbol{x}), \tag{A 2}$$

where the summation is over the complete set (except ψ_0), into (2.8), invoking $\mathscr{k}^2\psi_n = k_n^2\psi_n$, multiplying the result through by ψ_m, integrating over S, and invoking Green's theorem, we obtain the homogeneous set

$$\sum_n (S_{mn}-\kappa C_{mn})A_n = 0 \tag{A 3}$$

for the determination of the eigenvalues and the corresponding A_n, where

$$C_{mn} = k_n^{-2}\langle q_{mn}\cosh k_n h\rangle, \quad S_{mn} = k_n^{-1}\langle q_{mn}\sinh k_n h\rangle. \tag{A 4a, b}$$

The system (A 3) is the counterpart of (2.8). Multiplying it by the inverse matrix $[C_{mn}]^{-1}$ and introducing

$$[L_{mn}] = [C_{mn}]^{-1}[S_{mn}], \tag{A 5}$$

we obtain

$$\sum_n (L_{mn}-\kappa\delta_{mn})A_n = 0 \tag{A 6}$$

as the counterpart of (5.1). The corresponding counterpart of (6.3) is

$$\kappa = \sum_m\sum_n A_m L_{mn}A_n \Big/ \sum_n A_n^2. \tag{A 7}$$

The preceding results may be simplified by invoking (4.8) and expanding C_{mn}, S_{mn} and L_{mn} in powers of $\hat{h}$. Neglecting $O(\hat{h}^2)$ and invoking (A 1*b*), we obtain

$$C_{mn} = \delta_{mn} \cosh(k_n\langle h\rangle) + k_n^{-1}\langle q_{mn}\hat{h}\rangle \sinh(k_n\langle h\rangle), \tag{A 8a}$$

$$S_{mn} = \delta_{mn} k_n \sinh(k_n\langle h\rangle) + \langle q_{mn}\hat{h}\rangle \cosh(k_n\langle h\rangle), \tag{A 8b}$$

$$L_{mn} = \delta_{mn} k_n \tanh(k_n\langle h\rangle) + \langle q_{mn}\hat{h}\rangle \operatorname{sech}(k_m\langle h\rangle) \operatorname{sech}(k_n\langle h\rangle). \tag{A 9}$$

Substituting (A 9) into (A 7) and adopting the trial function $A_n = \delta_{1n}$, where $n = 1$ typically but not necessarily signifies the dominant mode, we obtain

$$\kappa = L_{11} = k_1 \tanh(k_1\langle h\rangle) + \langle q_{11}\hat{h}\rangle \operatorname{sech}^2(k_1\langle h\rangle), \tag{A 10}$$

which is equivalent to (4.10).

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